EFFECT OF THERMAL CONDUCTIVITY ON STRUCTURE AND CRITICAL PARAMETERS OF SHOCK WAVES IN PLASMA

Yu. A. Berezin and G. I. Dudnikova

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Results are presented for a numerical solution of the problem of shock-wave propagation in a cold, low-density plasma across a magnetic field with finite conductivity and electron thermal conductivity present; a comparison is made with results obtained from a solution without consideration of thermal conductivity. It is shown that the effect of thermal conductivity can be neglected for small Mach numbers ($M<2.5$). An isomagnetic density discontinuity is obtained for Mach numbers $2.8 \leqslant M \leqslant 3.3$. Increase in the magnetic field amplitude at the boundary of the plasma leads to a breakdown of the isomagnetic discontinuity. The critical Mach numbers which characterize the shock wave in this case are $M_* > 3.4$.

At the present time one should accept as proven the fact that there are critical Mach numbers M_* for which a qualitative change occurs in the structure of shock waves in a low-density plasma [1, 2]. As is well known, a similar phenomenon occurs in an ordinary thermally conducting gas – an isothermal density discontinuity. A careful study of stationary shock waves in a plasma without magnetic field but including Coulomb conductivity and thermal conductivity was made by V. S. Imshennik [3]. He showed that both continuous and discontinuous (isoelectron thermal discontinuity) solutions were possible depending on the velocity of the wave. Morton [4] studied stationary and nonstationary compression waves in a two-fluid plasma with magnetic field present; however, the effect of energy dissipation, which is necessary for creation of shock waves, was not considered in his paper and the effect of thermal conductivity was not taken into account. From an analysis of the singularities of the equations for the structure of a shock wave with conductivity and thermal conductivity present, Woods [5] found the critical parameters for which the solution became discontinuous. The authors investigated the problem of the structure of nonstationary and stationary shock waves in plasma across a magnetic field including consideration of conductivity and electron thermal conductivity.

1. SYSTEM OF EQUATIONS AND ITS STATIONARY SOLUTIONS

For one-dimensional motion of a two-fluid quasineutral plasma across a magnetic field including consideration of conductivity, dispersion, and electron thermal conductivity, we have the following system of equations:

$$\begin{gathered}
\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)=0, \qquad \frac{\partial}{\partial t}(\rho u)+\frac{\partial}{\partial x}\left(p+\frac{H^2}{8\pi}+\rho u^2\right)=0 \\
\frac{\partial}{\partial t}\left[\frac{3}{2}p+\frac{H^2}{8\pi}+\frac{1}{2}\rho u^2+\frac{m_i m_e c^2}{8\pi e^2\rho}\left(\frac{\partial H}{\partial x}\right)^2\right]+\frac{\partial}{\partial x}\left\{u\left[\frac{5}{2}p+\frac{1}{2}\rho u^2+\right.\right. \\
\left.+\frac{H^2}{4\pi}+\frac{m_i m_e c^2}{32\pi^2 e^2\rho}\left(\frac{\partial H}{\partial x}\right)^2\right]-\frac{c^2}{16\pi^2\sigma}H\frac{\partial H}{\partial x}-\chi\frac{\partial T}{\partial x}-\frac{m_i m_e c^2}{16\pi^2 e^2}H\left(\frac{\partial}{\partial t}+\right. \\
\left.\left.+u\frac{\partial}{\partial x}\right)\left(\frac{1}{\rho}\frac{\partial H}{\partial x}\right)\right\}=0, \qquad p=NT \\
\frac{\partial H}{\partial t}+\frac{\partial}{\partial x}\left\{uH-\frac{c^2}{4\pi\sigma}\frac{\partial H}{\partial x}-\frac{m_e m_i c^2}{4\pi e^2}\left(\frac{\partial}{\partial t}+u\frac{d}{du}\right)\left(\frac{1}{\rho}\frac{dH}{dx}\right)\right\}=0\,.
\end{gathered} \tag{1.1}$$

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Here, $\rho = N(m_i + m_e)$ is the plasma density, N is the number of particles of each kind per unit volume, σ is the plasma conductivity, and χ is the coefficient of electron thermal conductivity. The last two terms in the braces in the energy equation give the Joule heating of the electrons and the effect of thermal conductivity.

For solution of the stationary problem we shift to a coordinate system connected with the wave. We shall consider motion on a typical spatial scale which results from the finite conductivity but not from dispersion; we therefore neglect dispersion effects, which are proportional to electron mass. Using the continuity of mass and momentum flow and shifting to dimensionless variables, having selected as the scales for density, velocity, pressure, magnetic field, length, and collision frequency the quantities ρ_0,

$$V_A = H_0 / \sqrt{4\pi\rho_0}, \qquad H_0^2 / 4\pi, \qquad H_0$$

$$\delta = \frac{c^2}{4\pi\sigma_0 V_A}, \qquad \omega = \frac{eH_0}{\sqrt{m_i m_e} c}$$

($\sigma_0 = N_0 e^2 / m_e \nu$, ν = const is the effective collision frequency), we obtain

$$\frac{dH}{dx} = M\left(H - \frac{M}{u}\right)$$

$$\beta \frac{p - Mu}{u} \frac{du}{dx} = \frac{1}{2} u (5p + Mu + 2H^2) - uH\left(H - \frac{M}{u}\right) + \beta MH\left(H - \frac{M}{u}\right) - C \tag{1.2}$$

$$p = p_0 + 0.5 + M^2 - Mu - 0.5H^2$$

$$C = 0.5\ M\ (5p_0 + M^2 + 2).$$

Here M is the Mach number ($M = U/V_A$, where U is the wave velocity), u is the particle velocity in the wave system,

$$\beta = K\chi = \text{const}, \qquad K = \frac{\varkappa}{NN_0\omega\delta^2}; \qquad \varkappa = \frac{\nu}{\omega}.$$

Thus the problem of the structure of a stationary shock wave with finite conductivity and electron thermal conductivity present reduces to a system of two ordinary equations (1.2). The singularities 0 and 1 of system (1.2) correspond to the stationary states of the plasma ahead of the wave (unperturbed state) and behind the wave (perturbed state).

$$u_0 = M, \qquad H_0 = 1, \qquad p_0 \tag{1.3}$$

$$u_1 = \frac{M}{H_1}, \qquad M^2 = \frac{p_1 - p_0 + 0.5(H_1^2 - 1)}{1 - H_1^{-1}}, \qquad p_1 = \frac{p_0(4M - 1) + 0.5(H_1 - 1)^3}{4 - H_1} \tag{1.4}$$

Equation (1.4) is the Hugoniot condition for motion across a magnetic field.

As analysis shows, the solution in the case under consideration is discontinuous when the coefficient of the derivative goes to zero, i.e., $p_1 = Mu_1$, whence we obtain an equation for determination of the critical parameters M_* and H_* of a shock wave:

$$M_*^2 + p_0 - \frac{1}{2}(H_*^2 - 1) - \frac{2M_*^2}{H_*} = 0.$$

Under actual conditions $p_0 \ll 1$; therefore, using Eqs. (1.4), we obtain an equation for the critical amplitude of the magnetic field behind the wave front:

$$H_*^3 - 3H_*^2 + 2H_* - 6 = 0$$

whence it follows that $H_* = 3$.

Thus, with thermal conductivity and conductivity present, the profile of the shock wave is monotonic for $H_1 < H_* = 3$, $M < M_* = 3.46$ and discontinuous for $H_1 > 3$, $M > 3.46$. This result, of course, agrees with the result of [5], where it was obtained on the basis of classification of the singularities of the original equation system. Note that the value of the critical Mach number, $M_* = 3.46$, is independent of the specific form of the thermal conductivity coefficient.

If the effect of thermal conductivity is neglected, conductivity alone being present, the critical parameters of a shock wave are $H_* \approx 2.66$, $M_* \approx 2.76$.

We shall determine numerically a nonlinear solution of the problem of the structure of a stationary shock wave in a plasma with conductivity and thermal conductivity present. For this purpose it is necessary to make an analysis of the kinds of singularities 0 and 1 which correspond to the unperturbed and perturbed plasma states. Linearizing Eqs. (1.2) near the singularities, i.e., assuming that

$$u = u_{0,1} + u', \quad H = H_{0,1} + H' \quad (u' \ll u_{0,1}, \ H' \ll H_{0,1})$$

we obtain a characteristic equation, the roots of which for the unperturbed state are

$$k^{(0)} = \frac{3M}{4p_0}\left(1 \pm \sqrt{1 - \frac{8p_0}{3}\left(1 - \frac{1}{M^2}\right)}\right).$$

It then follows that the roots $k^{(0)}$, independently of Mach number $M > 1$, are real and of the same sign; therefore the singularity 0 is a node. The roots $k^{(1)}$ of the characteristic equation for the perturbed state depend on Mach number; for $M < 3.46$ and $H_1 < 3$, they are real and of opposite sign (singularity 1 is a saddle point), and for $M > 3.46$ and $H_1 > 3$, they are real and of the same sign (singularity 1 is a node). Thus when the shock-wave velocity is less than critical, the singularities 0 and 1 form a node-saddle-point pair as in an ordinary gas.

Proof of the existence and uniqueness of the solution in this case can be demonstrated as in [6], for example. When the shock-wave velocity is greater than critical, it is impossible to construct a profile which is continuous over all functions.

Using the analysis made above, a solution was obtained numerically for system (1.2) which gives the structure of a stationary shock wave.

Given below are the widths Δ_ρ of the particle density profile and Δ_H of the magnetic field profile for various Mach numbers.

H_1	2.60	2.70	2.80	2.90	2.93	2.95	2.98	2.99
M	2.66	2.83	3.02	3.23	3.29	3.34	3.41	3.44
Δ_ρ	0.46	0.34	0.24	0.16	0.14	0.12	0.10	0.09
Δ_H	0.90	0.80	0.70	0.60	0.58	0.56	0.55	0.53

As usual, the width of the front is determined from the expressions

$$\Delta_\rho = \frac{\rho_1 - 1}{|d\rho/dx|_{\max}}, \quad \Delta_H = \frac{H_1 - 1}{|dH/dx|_{\max}}.$$

As the Mach number increases, the density profile becomes increasingly steeper as compared to the magnetic field profile.

It is clear that the width of the density profile decreases by approximately a factor of 5 with a change in Mach number from 2.66 to 3.44 while the width of the magnetic field profile falls by a factor of approximately 1.5. The sharp increase in the slope of the density profile when $M \to M_* = 3.46$ is evidence that this profile tends to become discontinuous. Calculations for various thermal conductivity coefficients χ show that for an unchanged width of the magnetic field front the width of the density profile increases as the thermal conductivity coefficient increases. Thus, for the case $M = 3.44$, we have

$$\Delta\rho = 0.09 \text{ for } \beta = 2.0,$$
$$\Delta\rho = 0.18 \text{ for } \beta = 20.$$

2. NONSTATIONARY SOLUTIONS

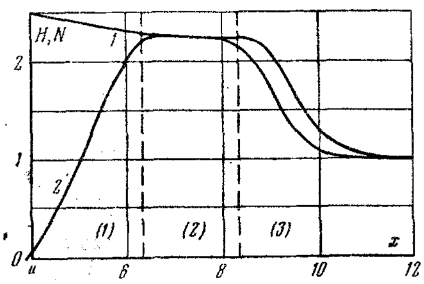

Fig. 1

The equation system (1.1), written in dimensionless variables and Lagrangian coordinates, was solved on a computer using a difference scheme of second-order accuracy. Assuming that at the initial time a uniform, quasineutral, cold plasma ($p_0 \ll H_0^2/8\pi$) of density N_0 occupies the region $0 \le x \le a$ (direction of unperturbed magnetic field coincides with the x axis), on the boundary of which the magnetic field is increasing in accordance with the expression $H = 1 + A(1 - e^{-\theta\tau})$ (θ is the frequency of the external field in units of ω; A is the amplitude of the external field in units of H_0), we write the following initial and boundary conditions for the solution of system (1.1):

$$H(0,\tau) = 1 + A(1 - e^{-\theta\tau})$$

$$p(0,\tau) = 0, \qquad \frac{\partial T}{\partial x}(0,\tau) = 0$$

$$\frac{\partial H}{\partial x}(a,\tau) = 0, \qquad \frac{\partial p}{\partial x}(a,\tau) = 0$$

$$u(a,\tau) = 0, \qquad H(x,0) = N(x,0) = 1$$

$$u(x,0) = p(x,0) = 0$$

Here, τ is time in units of $\varkappa/\omega$, T is electron temperature in units of $H_0^2/8\pi N_0$, and p is pressure in units of $H_0^2/8\pi$.

We turn to a discussion of the results obtained for a solution of the given problem for small Mach numbers $M < 2.5$. For amplitude values $1.5 \le A < 2$ at the boundary of the plasma the shock wave produced is quasistationary, i.e., from some point in time τ (τ depends on the boundary conditions for the magnetic field) the velocity and amplitude of the wave and the width of the front remain practically constant.

Note that the Mach number for a shock wave was determined from the points of maximum slope of the magnetic field, i.e., $M = (x_2 - x_1)/(\tau_2 - \tau_1)$, where x_1 and x_2 are the Eulerian coordinates of max $|\partial H/\partial x|$ in the shock front at the times τ_2 and τ_1.

Typical profiles of the magnetic field H (curve 1) and of particle density N (curve 2) for a quasistationary shock wave

$$(A = 1.5, \quad \theta = 0.2, \quad \varkappa = 10, \quad \beta = 2, \quad M \approx 2.2)$$

which are shown in Fig. 1 as a function of the Eulerian coordinates, have the following characteristic regions:

1. the piston region, which is associated with diffusion of the magnetic field into the plasma to a distance $\delta_1 \sim (c^2 t/4\pi\sigma)^{1/2}$; in this region there is a continuous transition from the magnetic field maximum at the boundary of the plasma to a value equal to the amplitude of the shock wave; the temperature behaves in a similar manner, and the density rises from zero to a maximum value;

2) the region of piston-shock-front transition, in which the magnetic field, temperature, and density are practically unchanged;

3) shock front with a width Δ equal to the dissipative dimension $\Delta \sim c^2/4\pi\sigma V_A$.

The density profile lags behind the magnetic field profile by a distance $\delta_2 \sim 0.6\delta \sim c^2/4\pi\sigma V_A(M-1)$, as should be expected for a resistive dissipative mechanism in the shock front. A characteristic feature of a quasistationary shock wave is the presence of region 2, which arises at the time the wave leaves the piston. Note that for large values of the external field amplitude (for example, $A = 10$, $\theta = 0.2$) a shock wave is not formed and "piling-up" of the plasma by the external magnetic field occurs.

Values of the electric fields and of the potential can be found from the equations of motion:

$$E_x = -\frac{m_i}{eN(m_i + m_e)}\frac{\partial}{\partial x}\left(p + \frac{m_i - m_e}{m_i}\frac{H^2}{8\pi}\right)$$

$$E_y = \frac{1}{c}\left(uH - \frac{c^2}{4\pi\sigma}\frac{\partial H}{\partial x}\right), \qquad \varphi = \int_x^a E_x\,dx\,.$$

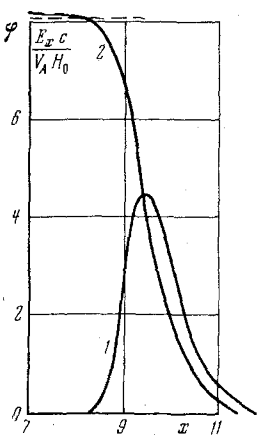

Fig. 2

The behavior of the y component of the electric field is similar to the behavior of the magnetic field; the x component of the electric field has a parabolic form with the maximum in the shock front since in precisely that place a large field is needed for realization of the quasineutrality and of the condition. The dependence of the x component of the electric field (curve 1) and of the potential (curve 2) on the Eulerian coordinates in a quasistationary shock wave with the parameters $A = 1.5$, $\theta = 0.2$, $\varkappa = 10$, and $\beta = 2$ is shown in Fig. 2. The dashed line gives the value of the potential φ_* behind a stationary shock front, which was obtained from the Hugoniot condition

$$\varphi_* = M^2\frac{\sqrt{m_i/m_e}}{2\varkappa}\left(1 - \frac{1}{H^2}\right)\frac{V_A H_0 \delta}{c}\,.$$

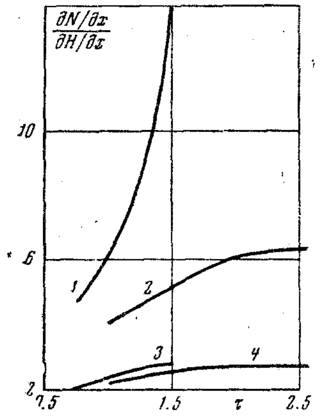

Fig. 3

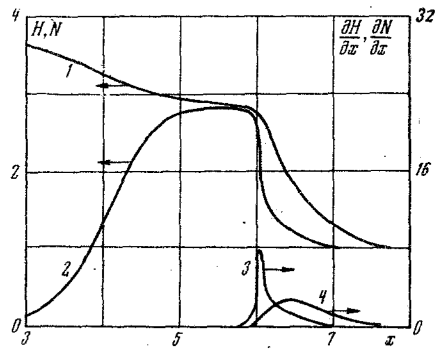

Fig. 4

A comparison of the solutions of the problem for small Mach numbers ($M < 2.5$) for thermal conductivity coefficient values $\chi = 0$ and $\chi \neq 0$ showed that inclusion of thermal conductivity leads only to an insignificant increase in the width of a quasistationary shock front (width of shock front for $\beta = 2$ was approximately 2% greater than width of the front for $\beta = 0$), i.e., the effect of thermal conductivity on the structure of the shock front is insignificant for small Mach numbers. The solution of the nonstationary problem for $M < 2.5$ is in good agreement with the stationary solution.

We consider the results for Mach numbers $M > 2.5$. For comparison we make a brief analysis without consideration of electron thermal conductivity ($\chi = 0$). Increase in the magnetic field amplitude at the plasma boundary leads to an increase in amplitude and velocity of the shock wave. In this case there is an enhancement of the effects of nonlinearity and nonstationarity, through which the structure of the shock front is qualitatively changed. Thus for an external magnetic field amplitude $A > 2.6$, the shock wave without consideration of thermal conductivity is nonstationary, i.e., its velocity and width of front change continuously with time. In addition, in contrast to the quasistationary mode, a continuous increase in the slope of the particle density profile occurs while there is an insignificant change in the width of the magnetic field front, i.e., the solution approximates a discontinuous solution. This happens for Mach numbers $M_* \gtrsim 2.8$.

Figure 3 shows the dependence of the maximum slope of the particle density profile $\partial N/\partial x$ (curves 1 and 2) and of the magnetic field (curves 3 and 4) in a shock wave with electron thermal conductivity and magnetic viscosity present for the case $A = 2.7$, $\varkappa = 10$, which is typical of Mach numbers $2.8 \lesssim M \lesssim 3.3$. Curves 1 and 3 correspond to the value $\beta = 0$, and curves 2 and 4 to the value $\beta = 2$. It is clear from the curves that from some point in time τ (τ depends on boundary conditions) the density profile can be characterized by an approximately constant width Δ_1; in this case the shock-wave velocity is practically unchanged. All this makes it possible to speak of a quasistationary mode in which there is compensation of the nonlinear broadening produced by thermal conductivity.

Figure 4 shows the spatial profile of the magnetic field, 1, of particle density, 2, of the density derivative, 3, and of the derivative of the magnetic field, 4, in the shock front during an isomagnetic discontinuity ($A = 2.7$, $\varkappa = 20$, $\beta = 2$, $M \approx 3.1$).

The width of the magnetic field front is considerably greater than the width of the density front (Fig. 4), i.e., a density discontinuity occurs during a practically constant magnetic field – an isomagnetic density discontinuity, which was obtained with external magnetic field amplitudes $2.7 \leq A \leq 4$ and shock-wave amplitudes $2.9 \lesssim H \lesssim 3.0$. With further increase in the external field ($A > 4$), a continuous rise in amplitude and velocity of the shock wave occurs because of the nonstationarity associated with the piston, which leads to breakdown of the shock wave.

Figure 5 shows typical spatial profiles of the magnetic field H (curves 1, 2, 3) and of the particle density N (curves 4, 5, 6) at successive times $\tau = 1.4, 1.6, 1.8$ for the nonstationary mode ($A = 8$, $\varkappa = 10$, $\beta = 2$).

Critical Mach numbers $3.4 \lesssim M_* \lesssim 3.8$, which characterize a shock wave with thermal conductivity and magnetic viscosity present at the time of breakdown, were obtained for magnetic field amplitudes $5 \leq A \leq 8$ at the plasma boundary; critical shock-wave amplitudes were $3 \lesssim H_* \lesssim 3.2$.

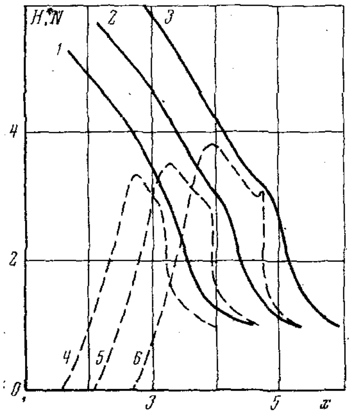

Fig. 5

We compare the particle velocity $|u-MV_A|$ behind the shock front (in the wave system) with the velocity of ion-acoustic waves $c_s = \sqrt{T/m_i}$.

For small Mach numbers, $c_s < |u-MV_A|$. As the shock-wave velocity increases, the ion sound velocity rises because the plasma temperature increases, and finally the velocity of ion sound becomes equal to the particle velocity relative to the shock front. For the stationary case this equality occurs for a magnetic field value behind the shock front equal to the critical value $H_* = 3$.

Thus the time of breakdown for a shock wave with finite conductivity and electron thermal conductivity present corresponds to the time of equalization of ion sound velocity and particle velocity behind the shock front.

From the solution of the nonstationary problem one can arrive at the following conclusions:

1) For small Mach numbers the shock wave is quasistationary and thermal conductivity leads to an insignificant increase in the width of the front.

2) A quasistationary isomagnetic density discontinuity is obtained for Mach numbers $2.8 \leqslant M \leqslant 3.3$ with electron thermal conductivity present.

3) Breakdown of shock waves in the presence of thermal conductivity is observed for Mach numbers $M_* > 3.4$.

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